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Propagation of weak shock waves through turbulence

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The effect of turbulence on the structure of weak shock waves is investigated. The equilibrium structure is shown to be governed by a balance between nonlinear steepening and the turbulent scattering of acoustic energy out of the main wave direction. The scattered energy appears as perturbations behind the shock front. For conditions typical of sonic booms in atmospheric turbulence the wave structure is governed by a Burgers equation similar to that describing viscous shocks, except that parameters related to the turbulence appear instead of the viscosity coefficient. The magnitude of the perturbations following a shock is estimated from first-order scattering applied to a thickened shock. Predictions of shock thicknesses and perturbations compare favourably with available experimental data. The approach used in the analysis of shock structure is to account for energy scattered from a single wave propagating a long distance through turbulence. This avoids difficulties of physical interpretation which arise if an ensemble-averaged structure is calculated, which is the usual approach in turbulent scattering analysis.

1. Introduction

The propagation of weak pressure waves through turbulence modifies the wave shape and structure compared with those for the non-turbulent case. These effects have been observed in sonic boom signatures and other weak shock waves. For example, sonic-boom theory for a stratified atmosphere predicts signatures consisting of very thin viscous shocks connected by slow pressure changes, usually expansions. However, experimental observations in the real turbulent atmosphere (Maglieri 1967, 1968; Garrick & Maglieri 1968; Reed 1969) usually exhibit two additional features.

Random perturbations and spikes are superimposed on the basic wave shape and the shock thicknesses are of order 10^3 times those predicted using ordinary viscosity and heat conduction. These effects are shown in figure 1. Note that except for the random perturbations and excessive shock thickness that the experiments agree quite well with the non-turbulent predictions. Tests made with microphones placed on tall towers and balloons (Maglieri 1967) have indicated that most of the perturbations of the wave shape originate in the lowest few thousand feet of the atmosphere. The magnitudes of these perturbations correlate qualitatively with the level of turbulence one would expect in the

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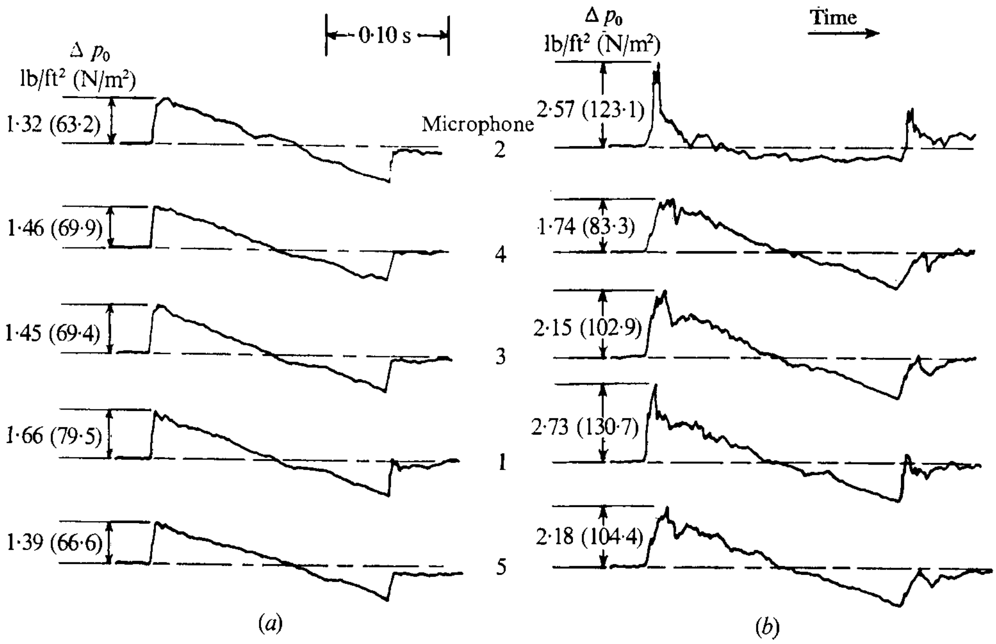

FIGURE 1. Typical flight test measurements for two different meteorological conditions. From Hilton, Huckel & Maglieri (1966). (a) Low wind velocity. (b) Strong gusty wind.

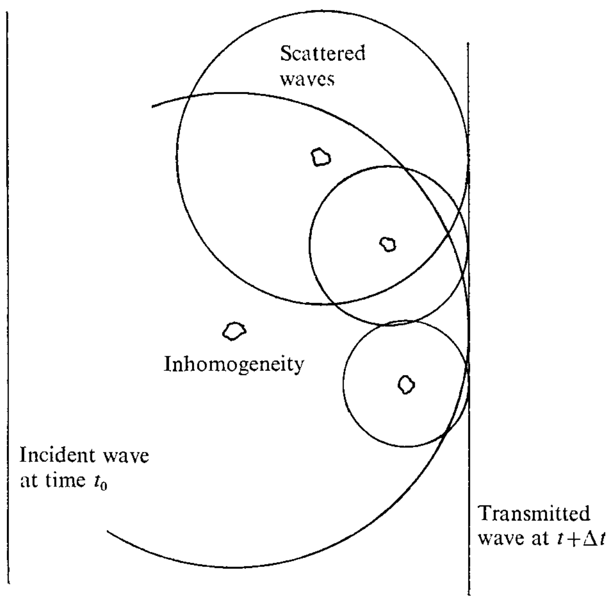

FIGURE 2. Schematic representation of scattering.

atmospheric boundary layer from the meteorological conditions during the tests. The perturbations behind the front and rear shocks of sonic-boom signatures are identical for a given signature. The time between the two shocks is small compared with time scales of atmospheric turbulence, so that both shocks encounter essentially the same turbulence.

The phenomena of thickening and perturbations are therefore associated with weak shock waves propagating through turbulence. In this paper, the problem of a weak plane shock wave propagating through homogeneous turbulence will be considered.

An appropriate method for examining weak waves in weak turbulence (atmospheric turbulence generally being weak) is scattering theory. Scattering theory is a perturbation scheme in which the strength of the turbulence, ϵ ($\epsilon \ll 1$), is used as the expansion parameter. A solution for the wave overpressure is sought in the form

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \dots . \tag{1}$$

The first term p_0 corresponds to propagation without turbulence and is called the incident wave. This does not quite satisfy the equations of motion at the inhomogeneities associated with the turbulence. The interaction of the incident wave with each inhomogeneity acts as an acoustic source, radiating a wave of strength ϵ which decays like $1/r$ (r = distance from the source). This is illustrated schematically in figure 2. The incident wave is shown as the straight line, and the circles represent the scattered waves from several sources. The contribution from each source, to first order in ϵ, is integrated over the turbulent region ('scattering volume') to give ϵp_1. Dropping terms of higher order than ϵp_1 is called the Born approximation.

In an analysis based upon first-order scattering by turbulence, Crow (1968, 1969) has explained many of the characteristics of the random perturbations of the observed wave. Crow modelled the incident shock as a step-function acoustic wave. His results therefore include scattering of very high frequency components in the assumed incident wave. First-order scattering theories based on a harmonic incident wave show that high frequencies are very strongly scattered at small scattering angles (Chernov 1960; Tatarski 1961; Batchelor 1957; Lighthill 1953). As a result, the generally reasonable values predicted by Crow's analysis become enormous near the shock front, whereas the actual observed perturbations reach a finite maximum. In order to predict the maximum mean-square perturbations correctly, the thickened shock structure must be included. Crow noted this and surmised that a second-order theory would be necessary to find the shock structure.

It is, indeed, necessary to use a second-order theory to explain shock thickening. First-order scattering is linear in the turbulence, which is taken to be random with zero mean, and any average of a first-order quantity is therefore zero. The phenomenon of thickening has a non-zero mean, as shocks are always observed to thicken relative to those for the non-turbulent case. Thus first-order scattering alone cannot account for thickening, and the scattering analysis must be carried through to at least second order.

Second-order scattering corresponds to the term $\epsilon^2 p_2$ in (1). It represents waves 'first scattered' from ϵp_1 plus any other second-order effects. In particular, the first scattered waves carry with them energy of order ϵ^2, so that the solution up to $\epsilon^2 p_2$ for the emergent wave in figure 2 must include a reduction in energy from that of p_0, the amount of reduction corresponding to the energy contained in ϵp_1. This dissipation is stronger for high frequencies, so that the high frequency components of the incident waves are more strongly attenuated.

In a previous paper by the authors (George & Plotkin 1971) in which only turbulent sound speed fluctuations were considered, thickening of the shocks was shown to be due to this scattering of the higher frequency components out of the incident wave direction. In the present paper the analysis is extended to include turbulent velocity and density fluctuations. A preliminary version of this extension was presented at the 1970 A.I.A.A. Aerospace Sciences Meeting (Plotkin & George 1970). The effect of shock thickness on the perturbations will also be briefly considered.

Because turbulent scattering is a random process, statistical results are usually sought, generally in the form of ensemble averages. This is the approach used by Howe (1971*a, b*) and Cole & Friedman (1971) in their investigations of shock structures in turbulence. However, the use of an ensemble average is not entirely satisfactory for describing shock waves; experimental observations are of individual waves, not ensemble averages. An ensemble average may display features which are never seen in an individual occurrence. For example, an ensemble average of coin tossing might lead us to believe that each side of a coin is half heads and half tails. In the case of shock waves, an ensemble average of waves with random arrival times may appear to have a thicker structure than any one wave (George 1971). A more satisfactory approach, used in this paper, is to base the decay of the incident wave on a balance of first scattered energy. In this way, the structure of an individual shock wave may be found. The energy-balance approach has the further advantage that energy scattered from the shock front is more readily related to the perturbations behind the front. This is important in determining when a thickened shock structure may exist. It will be shown that such a structure exists only when the propagation distance through turbulence is long enough for the first scattered energy (the perturbations) to have fallen behind the front.

For finite amplitude waves nonlinear effects must be included. These tend to make the wave steepen, thus counteracting the broadening of the wave due to the dissipation discussed above. In the next section an approximate partial differential equation governing the wave structure (less the first-order perturbations) is derived for conditions where dissipation due to scattering and weak nonlinear steepening is the dominant effect. The physical description of such a shock is analogous to that of a viscous shock except that here the dissipated energy is turned into random wave motion behind the shock (the perturbations), while in the viscous case the dissipated energy is turned into random molecular motion (heat) behind the shock.

2. Analysis

Consider small disturbances of a medium with constant mean properties and random fluctuations in sound speed, density and velocity (zero mean velocity). Pressure, density, velocity and sound speed are written as $p_\infty + p$, $\rho_\infty + \rho_T + \rho$, $\mathbf{u}_T + \mathbf{u}$ and $a_\infty + a_T + a$. The subscript ∞ denotes the undisturbed mean, the subscript T denotes random variations from the mean (turbulence) and unsubscripted quantities are perturbations associated with wave motion. Gravity is neglected and the turbulent Mach number $\langle|\mathbf{u}_T|\rangle/a_\infty$ is assumed small, so that $p_T \approx 0$. It is also assumed that the time scale of the turbulence is much larger than that of the wave motion, so that $\partial(\;)_T/\partial t$ may be taken as zero.

The mass and momentum equations for this system are

$$\partial p/\partial t + (\mathbf{u}_T + \mathbf{u}) . \nabla p + (\rho_\infty + \rho_T + \rho)(a_\infty + a_T + a)^2 \nabla . (\mathbf{u}_T + \mathbf{u}) = 0, \tag{2a}$$

$$\partial \mathbf{u}/\partial t + (\mathbf{u}_T + \mathbf{u}) . \nabla(\mathbf{u}_T + \mathbf{u}) + \nabla p/(\rho_\infty + \rho_T + \rho) = 0. \tag{2b}$$

The energy equation in the form $DS/Dt = 0$ has been used to eliminate ρ derivatives and the ideal gas law is assumed.

The turbulent quantities are taken to be

$$a_T = \epsilon a_\infty \mu, \quad \rho_T = \epsilon \rho_\infty \nu, \quad \mathbf{u}_T = \epsilon a_\infty \mathbf{U}, \tag{3a, b, c}$$

where μ, ν and $\mathbf{U}$ are random functions of space with zero means, and $\epsilon \ll 1$. The wave overpressure is taken to be of order δ relative to the ambient pressure, $\delta \ll 1$.

The gradient of (2*b*) may be subtracted from the time derivative of (2*a*) to give a single inhomogeneous wave equation. To third order in ϵ and δ, the wave equation may be written as

$$\frac{1}{a_\infty^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \Box^2 p = S_1 + N_2 + S_1 N_1 + S_2 + N_3 + \ldots, \tag{4}$$

where $S_1(p) = 2\epsilon\mu\nabla^2 p - \epsilon\nabla\nu . \nabla p + 2\epsilon\rho_\infty a_\infty \nabla . (\mathbf{U} . \nabla \mathbf{u}) = O(\delta)$,

$$S_2 = O(\delta\epsilon^2),$$

$$N_2(p) = -2\frac{\rho_\infty}{a_\infty}\frac{\partial}{\partial t}(a\nabla . \mathbf{u}) - \frac{\partial}{\partial t}(\rho\nabla . \mathbf{u}) - \frac{1}{a_\infty^2}\frac{\partial}{\partial t}(\mathbf{u} . \nabla p) + \nabla . (\rho\,\partial\mathbf{u}/\partial t) + \rho_\infty \nabla . (\mathbf{u} . \nabla u) = O(\delta^2),$$

$$S_1 N_2 = O(\delta^2\epsilon), \quad N_3 = O(\delta^3)$$

(see Plotkin 1971*a*).

To allow for the finite amplitude of the waves the expansion introduced as equation (1) is generalized to include δ as well as ϵ. If at a given time a portion of a plane wave is given by $p = \delta p_{10}(x - a_\infty t)$ the following expansion is assumed for the pressure a short time later:

$$p = \delta p_{10} + \delta\epsilon p_{11} + \delta\epsilon^2 p_{12} + \delta^2 p_{20} + \delta^2\epsilon p_{21} + \delta^3 p_{30} + \ldots. \tag{5}$$

Note that the indexing is such that p_{ij} is preceded by the factor $\delta^i\epsilon^j$.

It should be pointed out that (5) is merely an explicit statement of the expansion implicitly assumed in the usual acoustic scattering analyses. The usual expansion (1) is based on the acoustic equations, which are the first terms of an expansion in δ. Because higher-order terms in δ are important in the present problem, the two-parameter expansion is used explicitly.

On using (5) in the wave equation (4), and separating powers of ϵ and δ, the following set of wave equations is obtained:

$$\Box^2 p_{10} = 0, \quad \Box^2 p_{11} = S_1(p_{10}), \tag{6a,b}$$

$$\Box^2 p_{12} = S_1(p_{11}) + S_2(p_{10}), \quad \Box^2 p_{20} = N_2(p_{10}), \tag{6c,d}$$

$$\Box^2 p_{21} = S_1 N_2(p_{10}, p_{11}) + S_1(p_{20}), \quad \Box^2 p_{30} = N_3(p_{10}, p_{20}), \tag{6e,f}$$

and so on. The solutions to these first six equations represent the incident wave, first-order scattering, second-order scattering (which includes energy loss due to first scattering), lowest-order nonlinear steepening, combined scattering of lowest-order nonlinear effects with steepening of first-order scattered waves, and third-order nonlinear effects. The right-hand side of each equation depends only on the solutions to the equations preceding it, so (6) may be solved sequentially for a given incident wave δp_{10}. The solution to any inhomogeneous wave equation

$$\Box^2 p = Q(x, t)$$

is given by the retarded time integral (Born & Wolf 1964)

$$p(\mathbf{x}, t) = \frac{1}{4\pi}\iiint \frac{Q(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/a_\infty)}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'. \tag{7}$$

Thus, the solutions to (6) are known in principle. The solution for p is the superposition of these various solutions, with the series truncated at some point.

The series solution (5) is only valid when the various solutions to (6) are small compared to δp_{10}; this implies a limit on propagation distance. The solution for longer propagation distances can be obtained by re-initializing after a short distance: after obtaining the solution over a distance short enough for the perturbations about δp_{10} to be small, the solution emerging from this layer is used as δp_{10} for the next. This can be accomplished by seeking a differential form of (5), where the difference quotients are formed over a distance small enough for small perturbations to be valid. The conditions required for this successive layer concept to be valid will be examined.

The various solutions of (6) are considered to be written in terms of co-ordinates $\xi = x - a_\infty t$ and t fixed relative to the wave and are inserted into (5). It is desired to truncate the series at the earliest possible point at which it contains enough terms to describe the shock wave. In a steady or near steady wave, nonlinear steepening tends to make the wave thinner while dissipation tends to thicken it. The truncated series must contain these two effects. The lowest-order steepening term is $\delta^2 p_{20}$, and the first term representing dissipation is $\delta\epsilon^2 p_{12}$, which corresponds to second-order scattering. Further terms in (5) are smaller than one of these two. In a steady shock these two terms are balanced, so the series may be truncated at this point. In a shock wave which is not near its steady state these two terms are not of the same magnitude. Whichever is the larger will be the important one in determining the evolution of the shock structure, so the series may be truncated after that one. Retaining the other will do no harm, as it is of smaller magnitude, and further terms in the series are still not important. Retaining only the largest steepening term and the largest dissipation term is consistent with the method of derivation of the usual viscous shock structure, such as the analysis by Lighthill (1956).

As was pointed out above, $\delta\epsilon p_{11}$ is not explicitly involved in the shock structure. This term represents random perturbations superposed on the mean structure. It is desired to derive an equation describing p without considering these. Defining

$$P = p - \delta\epsilon p_{11}, \tag{8}$$

an equation for P will be derived. This approach of removing the first-order perturbations is used rather than ensemble averaging (noting that $\langle\delta\epsilon p_{11}\rangle = 0$) because measurements of shock thickness are for single shocks, not ensembles. Any variation in arrival time for different members of an ensemble (essentially a random walk about the mean arrival time) may result in an apparent thickness in an ensemble average which is larger than the thickness of any one shock. The calculation of such an apparent thickness, which is not physically real for an individual shock, is discussed in detail by George (1971). Making this distinction of calculating P rather than $\langle p\rangle$, and keeping track of the location of the scattered energy, is vital if we are to be fully justified in interpreting our final result as a decay. For example, in two recent papers, Howe (1971 *a*, *b*) has treated the passage of waves through turbulence by considering the ensemble average of the waves. His final result for short waves, applied to shallow-water bores, is essentially the same as our final result for shock structure. He also provides some justification for applying second-order scattering theory to cases where scattering is not weak. However, in our approach we have found certain important additional restrictions on the application of this structure. Also, in Howe's analysis of a stretched string, what he interprets as a decay in the ensemble average can be shown to be a random phase shift of the members of the ensemble making up the coherent wave, and not a decay for any one realization.

Truncating (5) after the fourth term, using (8) and taking the time derivative at constant ξ yields

$$\frac{\partial p}{\partial t} = \delta^2\frac{\partial p_{20}}{\partial t} + \delta\epsilon^2\frac{\partial p_{12}}{\partial t}. \tag{9}$$

The solutions $\partial p_{20}/\delta t$ and $\partial p_{12}/\partial t$ must now be found. The nonlinear steepening term $\partial p_{20}/\partial t$ may be found by applying the retarded time integral (7) to (6 *d*). Because p_{10} is a plane wave the integration is fairly straightforward, giving

$$\frac{\partial p_{20}}{\partial t} = -\frac{\gamma+1}{2\gamma}\frac{a_\infty}{p_\infty}p_{10}\frac{\partial p_{10}}{\partial \xi}. \tag{10}$$

Alternatively, the second-order correction of Whitham (1956) can be applied to the plane incident wave to give exactly the same result.

The second scattering term $\partial p_{12}/\partial t$ is the solution to (6 *b*) and (6 *c*). These are the usual equations for second-order acoustic scattering and are linear in ∂p_{10}. The solution will therefore be some linear operator $\mathscr{D}$, say, acting on ∂p_{10}. The scattering sources depend on the turbulence, so $\mathscr{D}$ depends on spatial location and the particular realization of the turbulence. Thus,

$$\delta\epsilon^2\,\partial p_{12}/\partial t = \mathscr{D}\,(\delta p_{10}(\xi), x, \text{realization}), \tag{11}$$

where 'x' and 'realization' reflect the dependence of $\mathscr{D}$ on the local turbulence.

It is not generally possible to find $\mathscr{D}$ as it stands because only the statistical properties of the turbulence are known. Therefore it is necessary to make some sort of approximation and seek a statistical result. If the scattering is weak enough, so that p changes little over a propagation distance of many macroscale lengths L_0, i.e. $\delta\epsilon^2\,\partial p_{12}/\partial t \ll \delta p_{10}\, a_\infty/L_0$, and a solution is sought for long propagation distances, then it is reasonable to approximate $\mathscr{D}$ by its spatial average with ξ fixed:

$$\mathscr{D} \cong \overline{\mathscr{D}} = \lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}\mathscr{D}(\delta p_{10}(\xi); x+x', \text{realization})\,dx'.$$

By averaging over the propagation distance we shall be averaging out the fluctuating parts of $\mathscr{D}$ but keeping any mean change in the wave shape. We next assume that the turbulence is locally homogeneous and that the members of the ensemble of realizations are statistically equivalent (regular) in order to make the ergodic-type hypothesis

$$\overline{\mathscr{D}} = \langle\mathscr{D}\rangle,$$

where $\langle\;\rangle$ denotes the ensemble average. Physically, this implies that the x-averaged energy scattered in one realization is equal to the average energy scattered in an ensemble of realizations. Thus,

$$\delta\epsilon^2\,\partial p_{12}/\partial t \cong \langle\mathscr{D}(\delta p_{10})\rangle. \tag{12}$$

Recall that $\delta p_{10} = P[1+O(\epsilon)+O(\delta)]$ and that $\delta^2\,\partial p_{20}/\partial t$ and $\delta\epsilon^2\,\partial p_{12}/\partial t$ are the highest-order terms being considered. The incident wave δp_{10} may therefore be replaced by P in (10) and (12) with no further loss in accuracy. Using these in (9) gives

$$\frac{\partial P}{\partial t} + \frac{\gamma+1}{2\gamma}\frac{a_\infty}{p_\infty}P\frac{\partial P}{\partial \xi} \cong \langle\mathscr{D}\,(P)\rangle. \tag{13}$$

Attention is now devoted to finding $\langle\mathscr{D}(P)\rangle$. As already explained, $\langle\mathscr{D}\rangle$ accounts for the loss of energy to first scattered waves. In addition, there may be dispersive or other effects which might not show up in an energy balance. These could only be found from a full second-order scattering analysis. George & Plotkin (1971) and Plotkin (1971*a*) have applied a second scattering solution due to Keller (1962) for the case of sound speed inhomogeneities only to show for that case, and for waves of thickness T small compared with the macroscale L_0, that dispersive and other effects were small compared to dissipation. Because of the similarity of the scattering sources, it is expected that this is also true for the present more general case, which includes scattering from density inhomogeneities and velocity turbulence. This greatly simplifies the calculation of $\langle\mathscr{D}\rangle$ for $T \ll L_0$. A full second-order analysis was not carried out as the dissipation alone can be obtained from a simple energy balance based upon well-known first-order scattering formulae. These are generally derived for single frequency harmonic waves. Since $\langle\mathscr{D}\rangle$ is a linear operator we may apply a single frequency analysis to each Fourier component of the shock structure. The form of $\langle\mathscr{D}\rangle$ will therefore first be found for a harmonic wave.

Consider a scattering volume which is a cube of side L, and a harmonic incident wave $B_0 \exp[ik(x-a_\infty t)]$ normal to one of the faces of the cube. The emergent wave from the opposite face is $B\exp[ik(x-a_\infty t)]$. The first scattered waves propagating in the direction defined by the unit vector $\hat{\mathbf{n}}$ are given by $\epsilon b(\hat{\mathbf{n}}) \times \exp[ik(\hat{\mathbf{n}}.\mathbf{x}-a_\infty t)]$. The energy flux of a harmonic wave is proportional to its amplitude squared. On balancing the mean energy flux into and out of the cube to second order in ϵ, one obtains

$$\langle |B_0|^2\rangle L^2 = \langle |B|^2\rangle L^2 + \int_S \epsilon^2 \langle |b|^2\rangle \hat{\mathbf{n}}.d\mathbf{s},$$

where $d\mathbf{s}$ is an element of the surface of the cube. The change in intensity of $B\exp[ik(x-a_\infty t)]$ over a distance L is thus

$$-\frac{\Delta\langle |B|^2\rangle}{L} = \frac{1}{L^3}\int_S \epsilon^2\langle |b|^2\rangle \hat{\mathbf{n}}.d\mathbf{s}.$$

The incident wave propagates for a time L/a_∞, in which it travels a distance L. Considering $B\exp[ik(x-a_\infty t)]$ to be written in the co-ordinates (ξ, t) fixed with respect to the wave, we have

$$\left(\frac{\partial\langle |B|^2\rangle}{\partial t}\right)_\xi = -\frac{a_\infty}{L^3}\int_S \epsilon^2\langle |b|^2\rangle \hat{\mathbf{n}}.d\mathbf{s}. \tag{14}$$

Since the fractional change in $\langle B\rangle$ is assumed small over the time L/a_∞,

$$\frac{\partial\langle B\rangle}{\partial t} = \frac{1}{2}\frac{\partial\langle |B|\rangle^2}{\partial t}.$$

The reduction in $\langle B\rangle$ is the decay identified with second scattering, given by the operator $\langle \mathscr{D}\rangle$. Thus, from (14),

$$\langle \mathscr{D}(B\,e^{ik\xi})\rangle = -\frac{a_\infty}{2L^3}\epsilon^2\left(\int_S \left\langle \left|\frac{b}{B}\right|^2\right\rangle \hat{\mathbf{n}}.d\mathbf{s}\right) B\,e^{ik\xi}. \tag{15}$$

The mean second scattering operator is thus given by this integral of the mean first scattered intensity.

First-order scattering is governed by (6a) and (6b). The solution to (6a) is the incident wave, here taken to be a plane wave travelling in the x direction. For the plane incident wave $\delta p_{10}(x-a_\infty t)$, equation (6$b$) becomes

$$\Box^2 p_{11} = 2\mu\frac{\partial^2 p_{10}}{\partial x^2} - \frac{\partial \nu}{\partial x}\frac{\partial}{\partial x}p_{10} + 2\psi\frac{\partial^2 p_{10}}{\partial x^2} + 2\frac{\partial\psi}{\partial x}\frac{\partial p_0}{\partial x}, \tag{16}$$

where $\psi = U_x$, the turbulent velocity component parallel to the incident wave direction. The first term on the right-hand side represents scattering by sound speed (index of refraction) variations, the second that by density variations and the third and fourth that by turbulent velocity fluctuations. Because of the similarity between the first and third, and the second and fourth terms, if the problem is solved first for sound speed and density scattering, velocity scattering can be added to the solution simply by letting $\mu \to \mu+\psi$ and $\nu \to \nu - 2\psi$. The equation to be solved is thus

$$\Box^2 p_{11} = 2\mu\frac{\partial^2 p_{10}}{\partial x^2} - \frac{\partial\nu}{\partial x}\frac{\partial p_{10}}{\partial x}. \tag{17}$$

This may be solved by applying the retarded time integral (7). The solution at a large distance from the scattering volume is well known. For homogeneous isotropic turbulence, using the notation of (14) and (15), this may be written as

$$\epsilon^2\langle|b|^2\rangle = \frac{\epsilon^2 B^2 L^3 k^4}{R^2 2k\sin\frac{1}{2}\theta}\bigg[\langle\mu^2\rangle\int_0^\infty N_{\mu\mu}(r)\sin(2kr\sin\tfrac{1}{2}\theta)\,dr + 2\langle\mu\nu\rangle\sin^2\tfrac{1}{2}\theta\int_0^\infty N_{\mu\nu}(r)\sin(2kr\sin\tfrac{1}{2}\theta)\,dr + \langle\nu^2\rangle\sin^4\tfrac{1}{2}\theta\int_0^\infty N_{\nu\nu}(r)\sin(2kr\sin\tfrac{1}{2}\theta)\,dr\bigg], \tag{18}$$

where R is the distance from the centre of the cube, θ is the scattering angle (angle between the scattered wave direction and incident wave direction) and k is the wavenumber of the incident wave. The correlation functions are defined by

$$N_{\mu\mu}(r) = \frac{\langle\mu(\mathbf{x}+\mathbf{r})\,\mu(\mathbf{x})\rangle}{\langle\mu^2\rangle},\quad N_{\mu\nu}(r) = \frac{\langle\mu(\mathbf{x}+\mathbf{r})\,\nu(\mathbf{x})\rangle}{\langle\mu\nu\rangle},\quad N_{\nu\nu}(r) = \frac{\langle\nu(\mathbf{x}+\mathbf{r})\,\nu(\mathbf{x})\rangle}{\langle\nu^2\rangle}. \tag{19}$$

The three correlation functions are used so that μ and ν need not be related to each other in any specific way. This is necessary when extending (18) to velocity turbulence by $\mu \to \mu+\psi$ and $\nu \to \nu - 2\psi$. With the addition of this generalization, the analysis leading to (18) follows that of Chernov (1960).

Equation (18) is valid only in the Fraunhofer zone $R \gg L^2/\lambda(\lambda = 2\pi/k)$. (Note, however, that the total energy is the same in the near or far field.) It also includes the assumptions of weak scattering, locally homogeneous isotropic turbulence and $L \gg L_0$. These three assumptions have already been used in approximating D by $\mathscr{D}$. Scattering intensities predicted by (18) have been experimentally verified by Kallistratova (1959) and Kallistratova & Tatarski (1960). The turbulence model employed therefore appears to be reasonable.

We evaluate the energy scattered out of the cube on a sphere of radius $R \gg L$ where the propagation direction of the first scattered waves is radially outward. Spherical co-ordinates R, θ, ϕ are adopted, ϕ being the azimuthal angle. Owing to isotropy there is no dependence on ϕ, and for large R

$$\hat{\mathbf{n}}.d\mathbf{s} = 2\pi R^2\sin\theta\,d\theta, \tag{20}$$

where θ varies from 0 to π. Substituting this and (18) into (15), noting the trigonometric relation $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and carrying out the integration over θ yields

$$\langle\mathscr{D}(Be^{ik\xi})\rangle = -\epsilon^2 a_\infty k^2\bigg\{\langle\mu^2\rangle\int_0^\infty N_{\mu\mu}(r)(1-\cos 2kr)\,dr + 2\langle\mu\nu\rangle\int_0^\infty\frac{N_{\mu\nu}(r)}{(2kr)^2}[2(kr)\sin 2kr - ((2kr)^2-2)\cos 2kr - 2]\,dr + \nu^2\int_0^\infty\frac{N_{\nu\nu}(r)}{(2kr)^4}[-(2kr)^4\cos 2kr + 4((3(2kr)^2-6)\cos 2kr + ((2kr)^2 - 6(2kr))\sin 2kr + 6)]\,dr\bigg\}Be^{ik\xi}. \tag{21}$$

Details of the first θ integral, the $\langle\mu^2\rangle$ part, may be found in Chernov (1960). The $\langle\mu\nu\rangle$ and $\langle\nu^2\rangle$ integrals were obtained by integrating by parts twice and four times respectively.

For large kL_0, $$\langle\mathscr{D}(Be^{ik\xi})\rangle \sim -\epsilon^2 a_\infty k^2\langle\mu^2\rangle L_0 B e^{ik\xi}, \tag{22}$$

where the macroscale L_0 is defined as

$$L_0 = \int_0^\infty N_{\mu\mu}(r)\,dr. \tag{23}$$

The assumption leading to (22) is that $kL_0 \gg 1$, i.e. the wavelength must be small compared with the turbulent macroscale. It should be noted that (22) takes the same form as for viscous dissipation, with a different coefficient, so that the shock structure will be qualitatively similar to the well-known Taylor shock structure, but will have a different (and, as will be seen, much larger) thickness. It should also be pointed out here that Lighthill (1953) obtained a result equivalent to (22) for the same limit of large k, except that isotropy was not required. When (22) applies, therefore, the turbulence need not be isotropic. For the present analysis, however, assuming isotropy made possible the calculation of the smaller terms in (21) so that their magnitude could be estimated.

It is straightforward, but lengthy, to apply (21) to each Fourier component of the shock structure $P(\xi, t)$. On doing so, the following is obtained:

$$\langle\mathscr{D}(P)\rangle = D_{\mu\mu}(P) + D_{\mu\nu}(P) + D_{\nu\nu}(P), \tag{24}$$

where

$$D_{\mu\mu}(P) = \epsilon^2\langle\mu^2\rangle L_0 a_\infty\left[\frac{\partial^2 P}{\partial\xi^2} - \frac{1}{4L_0}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty} N_{\mu\mu}(\tfrac{1}{2}\xi')\,P(\xi-\xi')\,d\xi'\right], \tag{25a}$$

$$\begin{aligned} D_{\mu\nu}(P) = 2\epsilon^2\langle\mu\nu\rangle a_\infty\Bigg[&-\frac{1}{4}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty} N_{\mu\nu}(\tfrac{1}{2}\xi')\,P(\xi-\xi')\,d\xi' \\ &-\frac{1}{2}\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\frac{N_{\mu\nu}(\tfrac{1}{2}\xi')}{\xi'}P(\xi-\xi')\,d\xi' - \frac{1}{2}\int_{-\infty}^{\infty}\frac{N_{\mu\nu}(\tfrac{1}{2}\xi')}{\xi'^2}P(\xi-\xi')\,d\xi'\Bigg], \end{aligned} \tag{25b}$$

$$\begin{aligned} D_{\nu\nu}(P) = \epsilon^2\langle\nu^2\rangle a_\infty\Bigg\{&-\frac{1}{4}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty} N_{\nu\nu}(\tfrac{1}{2}\xi')\,P(\xi-\xi')\,d\xi' - \frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\frac{N_{\nu\nu}(\tfrac{1}{2}\xi')}{\xi'}P(\xi-\xi')\,d\xi' \\ &-3\int_{-\infty}^{\infty}\frac{N_{\nu\nu}(\tfrac{1}{2}\xi')}{\xi'^2}[P(\xi-\xi') - P(\xi)]\,d\xi' - 6\int_\xi d\xi\int_{-\infty}^{\infty}\frac{N_{\nu\nu}(\tfrac{1}{2}\xi')}{\xi'^3}P(\xi-\xi')\,d\xi' \\ &-6\int_\xi d\xi\int_\xi d\xi\int_{-\infty}^{\infty}\frac{N_{\nu\nu}(\tfrac{1}{2}\xi')}{\xi'^4}[P(\xi-\xi') - P(\xi)]\,d\xi'\Bigg\}, \end{aligned} \tag{25c}$$

where $\int_\xi d\xi$ denotes the indefinite integral with respect to ξ. In atmospheric turbulence $\langle\mu^2\rangle$, $\langle\mu\nu\rangle$ and $\langle\nu^2\rangle$ are of comparable order and $N_{\mu\mu}$, $N_{\mu\nu}$ and $N_{\nu\nu}$ have similar behaviour. These factors will thus be treated as comparable in estimating the relative magnitudes of the above terms.

The second term of $D_{\nu\nu}$ and the first terms of $D_{\mu\nu}$ and $D_{\nu\nu}$ can be shown to be of order T/L_0 compared with the first term of $D_{\mu\mu}$ by interchanging the differentiation and integration, integrating by parts, and then finding upper bounds by using

$$\int_{-\infty}^{\infty} N'(\tfrac{1}{2}\xi')\,P'(\xi-\xi')\,d\xi' \leqslant |P'_{\max}| \int_{-\infty}^{\infty} |N'(\tfrac{1}{2}\xi')|\,d\xi'.$$

The remaining terms of $D_{\mu\nu}$ and $D_{\nu\nu}$ have been shown by Plotkin (1971*a*) to be of order T/L_0. Thus

$$\langle \mathscr{D}(P)\rangle = \epsilon^2 \langle \mu^2 \rangle L_0 a_\infty \frac{\partial^2 P}{\partial \xi^2}\left[1 + O\left(\frac{T}{L_0}\right)\right]. \tag{26}$$

The dispersive effects discussed by George & Plotkin (1971) and Plotkin (1971*a*) may be shown to be of the order T/L_0 by a similar approach. For $T/L_0 \ll 1$, only the leading term of $\mathscr{D}$ is important. This single term alone would be obtained by applying (22) to a thin shock. This is to be expected since a shock thickness T is described by Fourier components $k \approx 1/T$, so that the condition $T \ll L_0$ is equivalent to $k \gg 1/L_0$.

Typically, L_0 is of the order of hundreds of feet for atmospheric turbulence (Lumley & Panofsky 1964; Busch & Panofsky 1968), while observed shock thicknesses are from one to ten feet. Thus only the leading term of $\mathscr{D}$ need be used to calculate the structure of these shocks. With μ replaced by $\mu + \psi$ to include velocity turbulence (note that ν is no longer important when $T \ll L_0$), (13) becomes

$$\frac{\partial P}{\partial t} + \frac{\gamma+1}{2\gamma}\frac{a_\infty}{p_\infty} P \frac{\partial p}{\partial \xi} = \epsilon^2 \langle (\mu+\psi)^2 \rangle L_0 a_\infty \frac{\partial^2 P}{\partial \xi^2}. \tag{27}$$

Except that $\epsilon^2\langle(\mu+\psi)^2\rangle L_0 a_\infty$ replaces $\frac{2}{3}\gamma\eta$, where η = kinematic viscosity, this is the same Burgers equation as that governing weak viscous shocks. This is the same result as that obtained by George & Plotkin (1971) for sound speed fluctuations alone, except that $\mu + \psi$ appears here instead of μ. The effect on shock structure of velocity turbulence is thus, to this approximation, the same as that of sound speed variations.

3. Limiting assumptions

At this point we review the assumptions made in developing (27).

(i) In order to use the basic expansion in δ and ϵ we need δ and ϵ small enough so that successive terms in the series (5) decrease, allowing the sequential solution of (6). The analysis uses an average value of p_{12}; thus each layer over which the analysis is applied must be several correlation lengths long. At the same time, however, the total first scattered wave $\delta\epsilon p_{11}$ must be small compared with δp_{10} for the expansion in ϵ to be valid. Scattering is strongest for the highest frequency components of a wave. For a wave of thickness T the highest frequency components have wavenumber $k \approx 1/T$. From (22) the condition for weak scattering is thus seen to be

$$\epsilon^2 \langle (\mu+\psi)^2 \rangle L_0 L/T^2 \ll 1, \tag{28}$$

where here L is several times L_0 in order to allow the statistical treatment of the waves. The total propagation length is considered as being made up of a number of layers with 're-initializing' taking place after each layer, as was described previously.

(ii) We have modelled the turbulence as locally homogeneous and isotropic. On the largest scales atmospheric turbulence is neither homogeneous nor isotropic but the loss of accuracy in our case is expected to be small in the limit $L_0 \gg 1$, as was discussed following (23).

(iii) We have considered the averaged effect of second-order scattering, and have approximated this effect for $T/L_0 \ll 1$ using an energy balance verified for this limit in our earlier work treating sound speed fluctuations alone.

(iv) We have considered the variable P, which is the wave pressure less the first-order perturbations $\delta\epsilon p_{11}$. Since, as was just mentioned, we have also taken the average of the effect of $\delta\epsilon p_{12}$, our P should be considered as a mean shape taken over the propagation distance of the wave less the random perturbations. However, if the first-order perturbations are to be recognized in an experiment as being separate from the shock wave front they must have lagged far enough behind the initial wave for their phase to be distinct from the wave components making up the initial front. High frequency waves are scattered primarily at an angle to the incident wave of order $2(kL_0)^{-1}$. For waves propagating at such an angle to have lagged at least one half wavelength, one requires $\lambda L/L_0^2 \geqslant \pi^2$, where L is the propagation distance. Similar estimates can be made considering non-linear effects and are discussed by Plotkin (1971*a*).

If the scattered energy within some cone of angle θ_0 is not distinct this may be allowed for in the energy balance by using θ_0 instead of 0 for the lower limit in the integrations in (18). If this is carried through for the same approximations as those leading to (27) the following is obtained (Plotkin 1971*a*):

$$\frac{\partial P}{\partial t}+\frac{\gamma+1}{2\gamma}\frac{a_\infty}{p_\infty}P\frac{\partial P}{\partial \xi}=\frac{\epsilon^2\langle(\mu+\psi)^2\rangle}{2\sin\frac{1}{2}\theta_0}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty}N\left(\frac{\xi'}{2\sin\frac{1}{2}\theta_0}\right)P(\xi-\xi')\,d\xi. \tag{29}$$

This result is not directly applicable, as the choice of θ_0 is not fixed but depends on L, the total distance travelled from the scattering region. However, the behaviour of (29) is qualitatively similar to that of (27). This demonstrates that the physical description developed here is correct even when the approximations leading to the Burgers equation (27) are not strictly applicable. Shock structures will not be calculated from (29).

4. Application to sonic booms in the atmosphere

Before calculating shock thicknesses from this result it is necessary to see when the various assumptions made are applicable. Much of the current interest is in sonic-boom shock waves. Observed amplitudes are typically $\Delta p/p_\infty = 0{\cdot}5\times10^{-3}$ and measured shock thicknesses are about 1 to 10 feet. Atmospheric turbulence is generally inhomogeneous, varying strongly with height, and is perhaps most satisfactorily described by spectra or by structure functions. However, owing to the increased effectiveness of large eddies in scattering, the macroscale L_0 appears naturally in our results. The value of L_0 increases approximately linearly with altitude through the atmospheric boundary layer while the value of ϵ^2 decreases.

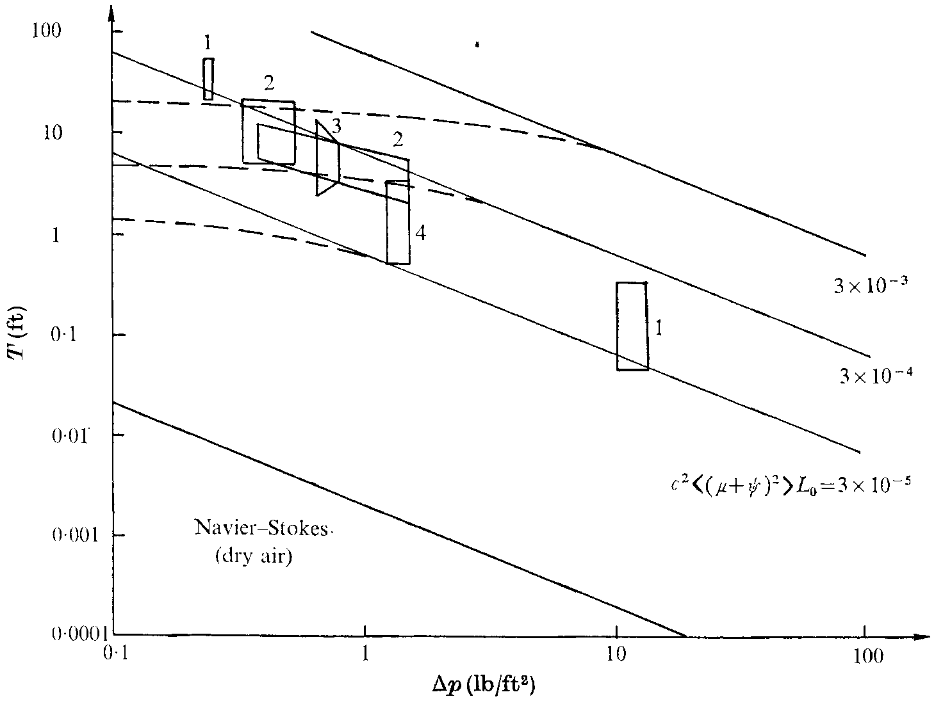

FIGURE 3. Comparison of predicted shock thicknesses with experimental data. 1, Reed (1969); 2, Maglieri (1968); 3, Maglieri, Huckel, Henderson & McLeod (1969); 4, Garrick & Maglieri (1968). ——, T_∞; - - -, T (3000 ft layer).

Using data presented by Busch & Panofsky (1968), Lumley & Panofsky (1964) and Tatarski (1961), as discussed by George & Plotkin (1971), we estimate typical turbulent daytime values of ϵ^2 to be in the range 10^{-7}–10^{-5} with L_0 ranging from several feet near the ground to values of the order of 10^3 ft near the top of the atmospheric boundary layer.

We now check the various assumptions:

(i) δ and ϵ are certainly small. Next, consider the condition (28) with L several times L_0. Noting that large $\epsilon^2\langle\mu^2\rangle L_0$ corresponds to large T, this result can generally be satisfied for the values of ϵ^2, L_0 and T_0 being considered.

(ii) The error due to the anisotropy and inhomogeneity of the atmosphere is expected to be small, particularly compared with that in estimating the turbulent parameters themselves.

(iii) Except for the last few hundred feet near the ground the condition $T \ll L_0$ is met. Experiments with microphones placed on towers (Magleiri 1967) indicate that sonic-boom signatures do not change appreciably in this last few hundred feet, so that the main effect of turbulence on sonic-boom signatures is at higher elevations where this condition is valid. This is also true for long-distance blast-wave propagation, where ray paths are refracted through high altitudes owing to the mean gradients in the atmosphere.

(iv) $\lambda L/L_0^2 \geqslant \pi^2$ requires quite large L when L_0 is large. By using a value of λ of order 3 ft, we see that for L_0 from, say, 3–300 ft the propagation distance L

must be greater than from 10 to 10^5 ft for most of the scattered energy to be distinct. Thus, for the larger values of L_0, equation (27) is based on an over-estimate of the scattered energy and will over-estimate the shock thickness. To determine the thickness accurately for this case would require the use of (29).

5. Calculation of shock thickness

The steady solution to the Burgers equation (27) is the classical Taylor solution, which gives a thickness based on maximum slope of

$$T = 16\frac{\gamma}{\gamma+1}\epsilon^2\langle(\mu+\psi)^2\rangle L_0\frac{p^\infty}{\Delta p} \tag{30}$$

for a shock of strength Δp.

Hopf (1950) and J. D. Cole (1951) have presented a closed-form solution to the time-dependent Burgers equation, given an initial profile. This may be used to calculate the shock structure when a thin viscous shock (approximately a step function) enters a region of turbulence. The distance required for the steady thickness given by (30) to develop may also be estimated. The question of distinctness becomes more critical for the high frequency components scattered from a thin shock, however, so the calculation of unsteady development must be regarded as approximate.

Figure 3 shows the steady-state shock thicknesses and the thickness after a 3000 ft layer for several values of $\epsilon^2\langle(\mu+\psi)^2\rangle L_0$, based on the values of ϵ^2 and L_0 discussed above, and the steady thickness of a Navier–Stokes viscous shock. Also shown are several groupings of shock thicknesses measured from sonic-boom and explosion experiments. The choice and interpretation of the data are discussed by George & Plotkin (1971). Although the exact values of $\epsilon^2\langle(\mu+\psi)^2\rangle L_0$ for the various experiments are not known, the present theory is seen to give the correct order-of-magnitude predictions for thicknesses, while the Navier–Stokes predictions are several orders of magnitude smaller.

6. Perturbations behind the shock

Crow (1968, 1969) has shown that for a step-function incident wave

$$\delta p_{10} = \Delta p H(\xi), \quad \text{where} \quad H(\xi) = \text{Heaviside step function},$$

the mean-square first-order fluctuations due to scattering by turbulence in the Kolmogorov inertial subrange are

$$\langle S_1^2\rangle = (h/h_c)^{-\frac{7}{6}}, \tag{31}$$

where $S_1 = \delta\epsilon p_{11}/\Delta p$, h = distance behind the shock front and h_c is a length which depends on the properties of the turbulence and the propagation path length through it. As was discussed by Crow, these perturbations become unreasonably large near the shock even if the viscous cut-off of the turbulent spectrum is accounted for. Enormous perturbations are predicted because of the very strong scattering of the high frequency Fourier components present in the discontinuous step-function incident wave. If a finite-thickness shock is considered then finite and reasonable maximum mean-square perturbations are predicted by Crows' theory.

Following Crow (1968), an incident wave of arbitrary structure can be represented as a sum of infinitesimal steps:

$$\begin{aligned}\delta p_{10}(\tau) &= \int d\delta p_{10}\, H[h-\tau(\delta p_{10})] \\ &= \int_{-\infty}^{\infty} \frac{d\delta p_{10}}{d\tau} H(h-\tau)\, d\tau \\ &= \int_{-\infty}^{h} \frac{d\delta p_{10}}{d\tau}\, d\tau. \end{aligned} \tag{32}$$

δp_{10} is identified with the present P, see George (1971). The first scattering response to a unit step $H(h)$ is $S_1(h)$, so

$$\delta p_0 + \delta \epsilon p_{11} = \int_{-\infty}^{h} [1-S(h-\tau)] \frac{d\delta p_{10}}{d\tau}\, d\tau,$$

which, on subtracting (32), becomes

$$\delta \epsilon p_{11} = \int_{-\infty}^{h} S_1(h-\tau) \frac{d\delta p_{10}}{d\tau}\, d\tau. \tag{33}$$

Taking the mean square of (33) gives

$$\langle(\delta\epsilon p_{11})^2\rangle = \int_{-\infty}^{h}\int_{-\infty}^{h} \langle S_1(h-\tau)\, S_1(h-\tau')\rangle \left(\frac{d\delta p_{10}}{d\tau}\right)\left(\frac{d\delta p_{10}}{d\tau}\right)' d\tau\, d\tau'. \tag{34}$$

To complete this calculation it is necessary to know the correlation

$$\langle S_1(h-\tau)\, S_1(h-\tau')\rangle,$$

which has not been found. However, it is possible to find an upper bound by noting that

$$\langle S_1(h-\tau)\, S_1(h-\tau')\rangle \leqslant \langle S_1^2(h-\tau)\rangle^{\frac{1}{2}} \langle S_1^2(h-\tau')\rangle^{\frac{1}{2}},$$

the expression on the right representing perfect correlation. Using this in (34) leads to

$$\langle(\delta\epsilon p_{11})^2\rangle^{\frac{1}{2}} \leqslant \int_{-\infty}^{h} \langle S_1^2(h-\tau)\rangle^{\frac{1}{2}} \frac{d\delta p_{10}}{d\tau}\, d\tau. \tag{35}$$

Using (31) for $\langle S_1^2\rangle$ gives

$$\langle(\delta\epsilon p_{11})^2\rangle^{\frac{1}{2}} \leqslant \int_{-\infty}^{h} \left(\frac{h-\tau}{h_c}\right)^{-\frac{7}{12}} \frac{d\delta p_{10}}{d\tau}\, d\tau. \tag{36}$$

Provided that $d\delta p_{10}/d\tau$ is finite, this integral converges. This upper bound for $\langle(\delta\epsilon p_{11})^2\rangle^{\frac{1}{2}}$ has been calculated for a shock of thickness $T = h_c$ whose structure is in accordance with the steady solution to the Burgers equation (27), and also for a ramp shock structure of the same thickness. The envelopes for both shocks are shown in figure 4, along with $(h/h_c)^{-\frac{7}{12}}$ for a step-function shock. At a distance of several T behind the shock the envelopes for both thickened shocks are close to $(h/h_c)^{-\frac{7}{12}}$, so (31) applies away from the shock front. Near the shock, the effect

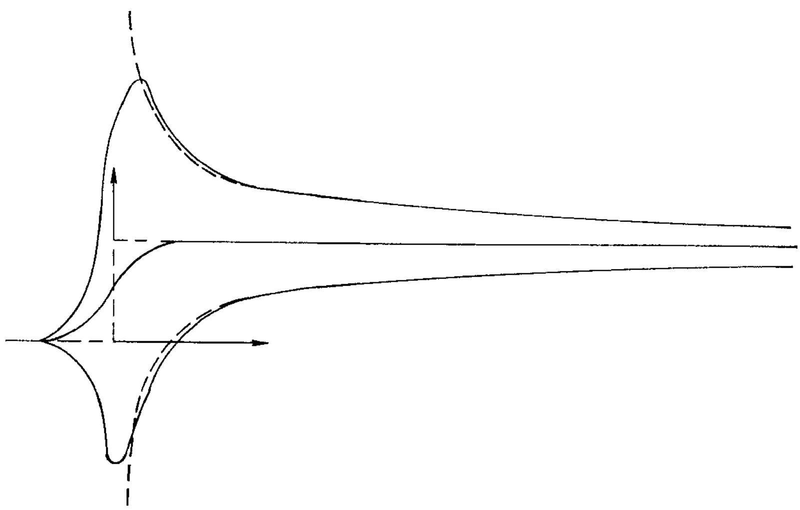

FIGURE 4. Root-mean-square perturbations for a thickened shock of thickness $T = h_c$, and $(h_c/h)^{\frac{7}{12}}$ for a step-function shock. ———, thickened shock; – – – –, step-function shock.

of finite thickness serves to 'cut off' equation (31). The details of the envelopes here differ for the two shock structures but the maximum perturbations are fairly close, so that it is the thickness and not the details of the shock structure that determines the magnitude of the largest perturbation. By scaling (36) it is easy to see that the maximum root-mean-square perturbation is of order $(h_c/T)^{\frac{7}{12}}$. Generally, $T > h_c$, so that perturbations are smaller than those shown in figure 4.

The above calculation is somewhat *ad hoc*, as it applies Crow's analysis to a thickened shock without relating h_c to T. The purpose was to show that the unreasonably large predictions near the shock front in Crow's analysis are due to his use of a zero-thickness shock. Because Crow's analysis applies only to a finite layer of turbulence (it is a first scattering analysis), a more proper approach would be to combine it with an unsteady thickening shock evolving from an initially thin viscous shock. Such a calculation has been performed by Cole & Friedman (1971) for typical atmospheric conditions, with good qualitative agreement with experimental boom measurements.

The physical model and analysis leading to the Burgers equation (27) predicts a steady shock structure after very long propagation paths through turbulence. This case is important for sonic booms near cut-off or blast waves near the ground. Calculation of perturbations behind a steady shock is difficult. Crow's first scattering analysis, even when applied to a thickened shock, diverges for long distances (Plotkin 1971*a*). For long distances, more terms in the expansion (5) must be retained. The next term to be examined is $\delta\epsilon^3 p_{13}$. This represents third-order scattering, or second scattering of $\delta\epsilon p_{11}$. Over long distances the first-order perturbations themselves are attenuated by scattering. This can be accounted for by applying the decay operator (21) or (24) to the first scattered waves. Such an analysis has been carried out with a number of simplifying approximations by Plotkin (1971*a*, *b*). The upper bound on the maximum perturbation behind a steady shock propagating through unbounded turbulence were found to be of the

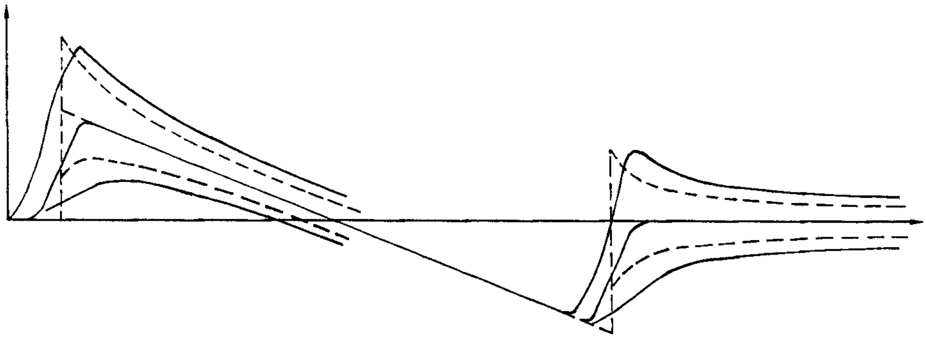

FIGURE 5. Root-mean-square perturbations on a 200 ft N-wave; $\Delta p/p_\infty = \frac{1}{2} \times 10^{-3}$; $L_0 = 100$ ft. ———, $\epsilon^2 = 10^{-5}$; ----, $\epsilon^2 = 10^{-7}$.

order of the shock strength, which is consistent with much of the experimental data. Figure 5, taken from Plotkin (1971*b*), shows the envelopes for the upper bounds of root-mean-square perturbations on an N-wave propagating through unbounded turbulence. That analysis included several approximations, the most severe of which was that all re-scattered perturbations were assumed distinct. However, the analysis provides a consistent physical description of the bounded perturbations behind a steady thickened shock.

7. Conclusion

The effects of turbulence on propagation of weak shock and other thin pressure waves under certain conditions have been shown to include a mean dissipative effect due to the wave energy scattered out of the incident wave direction. The scattered waves eventually become distinct in phase from the incident wave and appear as perturbations behind the initial front. Crow's method of calculating these perturbations is modified to account for the dissipative aspects of the scattering.

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